Suggested solutions to the IO (BSc) exam on June 14, 2011
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Question 1

a) **Solve for a subgame perfect Nash equilibrium of the model in which consumers with $v > a$, for some $a \in (0, 1)$, consume in period 1. Find the equilibrium value of a. Also identify the equilibrium values of p_1 and p_2.**

- We can solve the model by first studying the optimal behavior in period 2 for the firm and the consumers, given some arbitrary cut-off point $a \in (0, 1)$. Then, after having found the equilibrium value of p_2 as a function of a, we can study the optimal behavior in period 1, thereby identifying the equilibrium values of a and p_1.
- Remember that the monopoly firm is myopic — it cares only about the current period's profit when choosing the current period's price. The consumers, however, care about their future utilities — they use the (common) discount factor δ.

Second period

- Suppose consumers with $v > a$, for some $a \in (0, 1)$, consume in period 1.
  - The variable a is of course endogenous and we will later on determine its equilibrium value (in terms of exogenous parameters).
- In period 2, the monopolist then faces the demand schedule

$$q_2 = a - p_2.$$

  The derivation of this demand function makes use of the assumption that the v's are uniformly distributed on $[0, 1]$ and the fact that the remaining consumers in period 2 buy if and only if their valuation $v \in [0, a]$ exceeds the price p_2. (The students may want to draw a figure to illustrate how the demand function is obtained.)
- The price that maximizes period 2 profits, $\pi_2 = (a - p_2) p_2$, is

$$p_2 = \frac{a}{2}. \tag{1}$$

First period

- Given the period 1 price p_1 and the period 2 price $p_2 = \frac{a}{2}$, a consumer will consume in period 1 if and only if

$$v - p_1 \geq \delta (v - p_2) = \delta \left(v - \frac{a}{2} \right) \tag{2}$$

  Remember that a is defined as the value of v that makes the above inequality hold with equality:

$$a - p_1 = \delta \left(a - \frac{a}{2} \right) \Leftrightarrow a = \frac{2p_1}{2 - \delta}. \tag{3}$$

- The firm's profit at the stage when it chooses the period 1 price:

$$\pi_1 = [1 - a]\, p_1 = \left[1 - \frac{2p_1}{2-\delta}\right] p_1.$$

- FOC:

$$\frac{\partial \pi_1}{\partial p_1} = 1 - \frac{4p_1}{2-\delta} = 0$$

or

$$p_1^* = \frac{2-\delta}{4}. \tag{4}$$

Summing up

- By plugging (4) into (3), we can now get the equilibrium cut-off point

$$a^* = \frac{2p_1^*}{2-\delta} = \frac{1}{2}. \tag{5}$$

- By plugging (5) into (1), we get the equilibrium period 2 price

$$p_2^* = \frac{a^*}{2} = \frac{1}{4}.$$

- At the equilibrium we thus have

$$\boxed{p_1^* = \tfrac{2-\delta}{4} \qquad \text{and} \qquad p_2^* = \tfrac{1}{4},}$$

and half of the consumers consume in the first period ($\boxed{a^* = \frac{1}{2}}$).

b) **State the Coase conjecture. Explain the intuition.**

- The Coase conjecture concerns a situation where a monopoly firm, in each one of many periods, sells a good that is durable. The firm is allowed to choose a new price in each period. The fact that the good is durable means that those costumers who have bought the good will not need to purchase the good in any future period — these customers disappear from the demand. The Coase conjecture (it was later proven to, under certain conditions, hold as a result) states that:
  - When the length between time periods become smaller (or, equivalently, when the consumers' discount factor approaches one), the monopolist's profit converges to the marginal cost — the firm loses all its market power.

- The reason why this happens is that for any given price in a period, the consumers who find it worthwhile to purchase will be those with the highest valuation. That means that in the next period, those high-valuation consumers are not part of demand and therefore the optimal monopoly price must be lower (since demand is lower). In other words, if the monopoly firm cannot precommit to some sequence of prices but is optimizing in each period given the current demand, the price will gradually drop. However, if the consumers understand this they should have an incentive to wait with purchasing until a later period when the price has fallen. The only thing that may stop the consumers from waiting is that they are impatient and prefer immediate consumption to later, all else being equal. But if the length of time between periods is small or if the consumers are not very impatient (which is the condition in the conjecture), then the consumers don't mind waiting until the price has dropped. If so, the firm may be better off lowering the price straight ahead, so that it doesn't have to wait so long for its (perhaps small) profits.

- To further clarify the explanation we can relate to the result we obtained under a). In that model, whereas the second-period price is constant, *the first-period price is decreasing in the patience parameter δ.* This result is in the spirit of the Coase conjecture, although the monopolist in this simple example doesn't lose all its market power, only some of it.

c) **Define the "Herfindahl index" and the "3-firm concentration ratio". Also, consider a market with seven firms. Their market shares are 5, 5, 10, 10, 20, 20 and 30 percent. Calculate the Herfindahl index and the 3-firm concentration ratio for this market.**

- The Herfindahl index is defined as the sum of the squared market shares, $HI = \sum_{1=1}^{n} s_i$, where s_i is firm i's market share and n is the number of firms in the market.

  - Therefore, the Herfindahl index for this market equals

$$
\begin{aligned}
HI &= 2 \times \left(\frac{5}{100}\right)^2 + 2 \times \left(\frac{10}{100}\right)^2 + 2 \times \left(\frac{20}{100}\right)^2 + \left(\frac{30}{100}\right)^2 \\
&= \frac{50}{10,000} + \frac{200}{10,000} + \frac{800}{10,000} + \frac{900}{10,000} = \frac{1,950}{10,000} \\
&= 0.195.
\end{aligned}
$$

- The 3-firm concentration index ratio is defined as the sum of the three largest firms' market shares.

  - Therefore this ratio equals $0.3 + 0.2 + 0.2 = 0.7$.

Question 2

a) Solve for all subgame-perfect Nash equilibria of the game described above (however, do not bother about the mixed-strategy equilibrium at stage 1).

We can solve for the subgame-perfect Nash equilibria by using backward induction, i.e., by solving the game from the end. At stage 3 the customers are making their consumption decisions and that behavior is already summarized in the question. At stage 2 we are in one of four subgames, depending on the firms' choices at stage 1. In terms of the notation introduced in the question, the four subgames are $(x_1, x_2) \in \{(n,n), (d,d), (d,n), (n,d)\}$. For each subgame we must calculate the equilibrium prices and the equilibrium profits. However, the amount of calculations that we must do will not be that large, as we only need to investigate one of the subgames (d,n) and (n,d) (due to symmetry of the model) and (as will be explained below) the subgames (n,n) and (d,d) are also very similar.

The stage 2 subgame where neither discriminates: $(x_1, x_2) = (n, n)$

Firm 1's profits are

$$\Pi^1(p_1, p_2) = p_1 D_1(p_1, p_2) = p_1 \overline{\theta} = p_1 \left[\frac{p_2 - p_1 + 1}{2} \right].$$

The FOC is:

$$\frac{\partial \Pi^1(p_1, p_2)}{\partial p_1} = \left[\frac{p_2 - p_1 + 1}{2} \right] - p_1 \left[\frac{1}{2} \right] = 0.$$

By symmetry of the game, we obtain the Nash equilibrium of the subgame (n, n) by setting $p_1 = p_2 = p_{n|n}$ in this FOC. Doing that yields

$$\left[\frac{p_{n|n} - p_{n|n} + 1}{2} \right] - p_{n|n} \left[\frac{1}{2} \right] = 0 \Rightarrow p_{n|n} = 1.$$

Next, we get the firms' profits at the subgame (n, n) by setting $p_1 = p_2 = p_{n|n} = 1$ in the objective function:

$$\Pi^1\left(p_{n|n}, p_{n|n}\right) \equiv \Pi_{n|n} = p_{n|n} \left[\frac{p_{n|n} - p_{n|n} + 1}{2} \right] = \frac{1}{2}.$$

The stage 2 subgame where both discriminate: $(x_1, x_2) = (d, d)$

The profit functions at this subgame are

$$\Pi^1(p_1, p_2) = p_1 D_1(p_1, p_2) = p_1 (1 - \gamma) \overline{\theta} \qquad \text{and} \qquad \Pi^2(p_1, p_2) = p_2 D_2(p_1, p_2) = p_2 (1 - \gamma)\left(1 - \overline{\theta}\right).$$

Since these are exactly as in the (n, n) subgame but with each profit function being multiplied by $(1 - \gamma)$, the equilibrium prices are not affected: Both firms charge the price $p_{d|d}$, where

$$p_{d|d} = p_{n|n} = 1.$$

We get the firms' profits at the subgame (d, d) by setting $p_1 = p_2 = p_{d|d} = 1$ in the objective function:

$$\Pi^1\left(p_{d|d}, p_{d|d}\right) \equiv \Pi_{d|d} = (1 - \gamma)\, \Pi_{n|n} = \frac{1 - \gamma}{2}.$$

The stage 2 subgame where firm 1 only discriminates: $(x_1, x_2) = (d, n)$

Firm 1's profits are

$$\Pi_1(p_1, p_2) = p_1 D_1(p_1, p_2) = p_1 \overline{\theta} = (1-\gamma) p_1 \left[\frac{p_2 - p_1 + 1}{2} \right].$$

Firm 2's profits are

$$\Pi_2(p_1, p_2) = p_2 D_2(p_1, p_2) = p_2 \left[1 - (1-\gamma) \overline{\theta} \right] = p_2 \left[1 - \frac{(1-\gamma)(p_2 - p_1 + 1)}{2} \right]$$

Firm 1's FOC:

$$\frac{\partial \Pi^1(p_1, p_2)}{\partial p_1} = (1-\gamma) \left[\frac{p_2 - p_1 + 1}{2} \right] - (1-\gamma) p_1 \left[\frac{1}{2} \right] = 0 \Leftrightarrow 2p_1 - p_2 = 1 \quad (6)$$

Firm 2's FOC:

$$\frac{\partial \Pi^2(p_1, p_2)}{\partial p_2} = \left[1 - \frac{(1-\gamma)(p_2 - p_1 + 1)}{2} \right] - p_2 \left[\frac{1-\gamma}{2} \right] = 0 \Leftrightarrow 2p_2 - p_1 = \frac{1+\gamma}{1-\gamma} \quad (7)$$

Using (6) in (7) we get

$$2(2p_1 - 1) - p_1 = \frac{1+\gamma}{1-\gamma} \Leftrightarrow 3p_1 = \frac{1+\gamma}{1-\gamma} + 2 = \frac{3-\gamma}{1-\gamma} \Leftrightarrow p_1 = p_{d|n} = \frac{3-\gamma}{3(1-\gamma)},$$

which plugged back into (7) yields

$$2p_2 - \frac{3-\gamma}{3(1-\gamma)} = \frac{1+\gamma}{1-\gamma} \Rightarrow p_2 = p_{n|d} = \frac{6-2\gamma}{6(1-\gamma)} = \frac{3+\gamma}{3(1-\gamma)}.$$

The difference between the prices is

$$p_{n|d} - p_{d|n} = \frac{3+\gamma}{3(1-\gamma)} - \frac{3-\gamma}{3(1-\gamma)} = \frac{2\gamma}{3(1-\gamma)},$$

Therefore firm 1's profit is

$$\begin{aligned}
\Pi_1\left(p_{d|n}, p_{n|d}\right) &\equiv \Pi_{d|n} = (1-\gamma) p_{d|n} \left[\frac{p_{n|d} - p_{d|n} + 1}{2} \right] = (1-\gamma) \frac{3-\gamma}{3(1-\gamma)} \left[\frac{\frac{2\gamma}{3(1-\gamma)} + 1}{2} \right] \\
&= \frac{3-\gamma}{6} \left[\frac{2\gamma}{3(1-\gamma)} + \frac{3(1-\gamma)}{3(1-\gamma)} \right] = \frac{(3-\gamma)^2}{18(1-\gamma)}
\end{aligned}$$

Firm 2's profit is

$$\begin{aligned}
\Pi_2\left(p_{d|n}, p_{n|d}\right) &\equiv \Pi_{n|d} = p_{n|d} \left[1 - \frac{(1-\gamma)\left(p_{n|d} - p_{d|n} + 1\right)}{2} \right] \\
&= \frac{3+\gamma}{3(1-\gamma)} \left[1 - \frac{(1-\gamma)\left(\frac{2\gamma}{3(1-\gamma)} + 1\right)}{2} \right] \\
&= \frac{3+\gamma}{3(1-\gamma)} \left[1 - \frac{\left(\frac{2\gamma + 3(1-\gamma)}{3}\right)}{2} \right] = \frac{3+\gamma}{3(1-\gamma)} \left[1 - \frac{3-\gamma}{6} \right] = \frac{(3+\gamma)^2}{18(1-\gamma)}
\end{aligned}$$

The stage 1 game

Summing up, we thus have

$$\Pi_{n|n} = \frac{1}{2}, \qquad \Pi_{d|d} = \frac{1-\gamma}{2}, \qquad \Pi_{d|n} = \frac{(3-\gamma)^2}{18(1-\gamma)}, \qquad \Pi_{n|d} = \frac{(3+\gamma)^2}{18(1-\gamma)}.$$

It will be useful for the remaining analysis to relate the four profit levels to each other. First, by inspection it is obvious that $\Pi_{d|d} < \Pi_{n|n}$. Moreover, we have

$$\Pi_{n|n} < \Pi_{d|n} \Leftrightarrow \frac{1}{2} < \frac{(3-\gamma)^2}{18(1-\gamma)} \Leftrightarrow 9(1-\gamma) < (3-\gamma)^2 = 9-6\gamma+\gamma^2 \Leftrightarrow 0 < 3\gamma+\gamma^2,$$

which always holds. Finally, for positive values of γ it is clear from inspection that $\Pi_{d|n} < \Pi_{n|d}$. Overall we therefore have the relationships

$$\boxed{\Pi_{d|d} < \Pi_{n|n} < \Pi_{d|n} < \Pi_{n|d}.} \qquad (8)$$

We have now solved all the stage 2 subgames and derived expressions for the equilibrium profit levels in all of these. Using these profit levels we can illustrate the stage 1 interaction between the firms in a game matrix (where firm 1 is the row player and firm 2 is the column player):

	$x_2 = d$	$x_2 = n$
$x_1 = d$	$\Pi_{d|d}$, $\Pi_{d|d}$	$\Pi_{d|n}$, $\Pi_{n|d}$
$x_1 = n$	$\Pi_{n|d}$, $\Pi_{d|n}$	$\Pi_{n|n}$, $\Pi_{n|n}$

Inspecting the table, using (8), we see that there are two pure strategy Nash equilibria of this game, $(x_1, x_2) = (d, n)$ and $(x_1, x_2) = (n, d)$.

- **Conclusion:** the overall game has two SPNE (where the firms play pure at stage 1). In these equilibria, one of the restaurants discriminates whereas the other one does not.

b) Interpret your results: what is the economic logic that explains why the restaurants at stage 1 make the choices they make in the equilibria that you derived? When explaining that logic, make sure you answer the following two questions: (i) At stage 2, are the restaurants' choice variables strategic substitutes or strategic complements, and what is the significance of this? (ii) What is the significance of the assumption that each firm can observe the other firm's decision whether to discriminate before choosing the price at stage 2?

To understand the logic, suppose (to start with) that firm 1 expects firm 2 *not* to discriminate. Given that, what would be the consequences for firm 1's profit if firm 1 discriminated? We should expect there to be two effects:

1. A direct negative effect on firm 1's demand and therefore on firm 1's profit. If firm 1 refuses to sell to the minority customers, then it cannot earn any profits on those customers.

2. An indirect, strategic effect, which is positive: If firm 1 refuses to sell to the minority customers, then (by assumption) this choice is observed by firm 2 before firm 2 chooses its price. Moreover, firm 2's demand will go up, because those customers who are not served by firm 1 will go to firm 2 instead. The optimal response to an increase in the demand is to charge a higher price,[1] so by choosing $x_1 = d$ firm 1 can make p_2 go up. The fact that firm 2 charges a relatively high price is good for firm 1's profits, for this makes it possible for firm 1 to charge a relatively high price itself without losing too many customers to firm 2.

The fact that firm 1 discriminates thus leads to a loss in sales for firm 1, which is bad for profits (the negative direct effect). However, it also leads to a higher price for firm 1, which is good for profits (the positive strategic effect). The algebra under a) shows that, perhaps surprisingly, the strategic effect is so strong that also the overall effect is positive.

Key to the result is thus the strategic effect. For that effect to be present it is clear from the above explanation that firm 2 must be able to *observe* firm 1's (irreversible) decision to discriminate. What's important for firm 1 is that firm 2 *believes* that firm 1 discriminates, so that firm 2 has an incentive to raise its price (if firm 1 could fool firm 2 by pretending to discriminate but then not actually doing it, then that would be ideal for firm 1). If firm 2 observes firm 1's decision to discriminate (and knows that it's irreversible), then firm 2 will of course (correctly) believe that firm 1 discriminates.

In the stage 2 game the firms' choice variables (i.e., the prices) are strategic complements. This is crucial for the strategic effect to work in the right direction (i.e., for the effect to have a *positive* impact on firm 1's profit). To see this, note that for discrimination to have any chance of being profitable for firm 1, it must be that firm 1 optimally is charging a higher price with discrimination that without.[2] For that to be the case, the choice variables must be strategic complements, as illustrated by the following chain of reactions:

$$\boxed{\text{Firm 1 discriminates}} \overset{\text{a)}}{\Rightarrow} \boxed{\text{firm 2's demand} \uparrow} \overset{\text{b)}}{\Rightarrow} \boxed{p_2 \uparrow} \overset{\text{c) Due to strat compl.}}{\Rightarrow} \boxed{p_1 \uparrow}.$$

As long as the effects a) and b) work in the directions indicated above — which we should expect to be the case under quite general assumptions — strategic complements are required for firm 1's price to increase.

Finally we can also consider the possibility that firm 1 expects firm 2 to indeed discriminate itself. In this case, if firm 1 also discriminated then this would as before lead to a loss in sales for firm 1, which is bad for profits. Moreover, given that firm 2 also is not serving the minority customers, there

[1] Saying that firm 2's demand goes up and that it is this that makes firm 2 charge a higher price is a slight simplification. In fact it is not only that firm 2's demand goes up, but also that the own price elasticity of firm 2's demand goes down (because some of the customers can only buy from firm 2). It is the lower elasticity that makes firm 2 charge a higher price. If the demand increased but the elasticity remained the same, then this would not change firm 2's optimal price.

[2] At least that must be the case as long as the net demand effect for firm 1 is negative. In principle one could imagine that, even though firm 1 loses demand by not serving the minority customers, the fact that firm 2 raises its price could lead to a gain in demand for firm 1 that exceeds the loss it made by discriminating. However, it seems very unlikely that the indirect price effect can be that strong (and one can probably show that this is indeed impossible, due to the stability assumptions that will be satisfied in a standard Hotelling model like this one).

would not be any demand increase for firm 2 and therefore no strategic effect that could boost firm 1's profits. Therefore, if firm 1 expects firm 2 to discriminate, then we should not expect firm 1 to have an incentive to discriminate too. This observation, together with the ones we made above, help us understand why there can be an equilibrium where one firm discriminates, but not one where both firms do it simultaneously.